Lecture 20

Elementary Counting Problems

Permutations

Let us assume that n patients arrived at a dentist's office at the same time.
The dentist asks them to form an order and then he will treat them one by one.

How many different orders are possible? $n \times (n-1) \times (n-2) \times \ldots \times 1 = n!$

Definition: The arrangement of different objects into a linear order using each object exactly once is called a **permutation** of these objects.

Theorem: The number of all permutations of an n element set is $n!$.

containing unique elements

Permutations

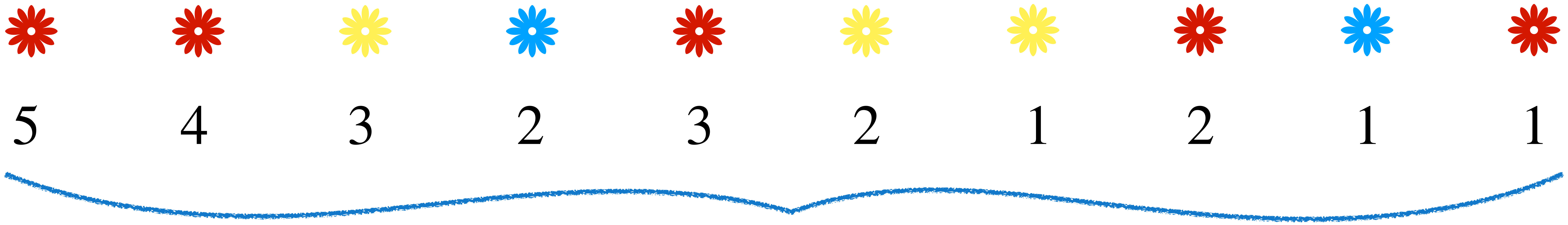
Example: A gardener has 5 red flowers, 3 yellow flowers and 2 blue flowers to plant in a row. In how many different ways can she do that?

Solution: The issue is that the objects are not all distinct.

Assume the gardener plants all the flowers and then labels red flower from 1 to 5, yellow flowers from 1 to 3, blue flowers from 1 to 2.

With labels the row of flowers can look in $10!$ different ways.

Consider the following arrangement of unlabelled flowers:



Possible number of labels.

Permutations

The red, yellow, and blue flowers can be labeled in $5!$, $3!$, and $2!$ ways, respectively.

Let x be the number of ways gardener can plant flowers without labels.

Then, $x \times 5! \times 3! \times 2! = 10! \implies x = \dfrac{10!}{5!.3!.2!}$ ■

Theorem: Let $n, k, a_1, a_2, \ldots, a_k$ be non-negative integers satisfying $a_1 + a_2 + \ldots + a_k = n$. Consider a multiset of n objects, in which a_i objects are of type i, for all $i \in [k]$. Then the number of ways to linearly order these objects is,

$$\frac{n!}{a_1!\,.\,a_2!\ldots a_k!}$$

Sequences

Theorem: The number of k-length sequence one can form over an n-element set is n^k.

Proof: We can choose the 1st element of the sequence in n different ways.

Similarly, the 2nd, 3rd, and other elements can also be chosen in n different ways.

Therefore, the total number of choices $= \underbrace{n \times n \times \ldots \times n}_{k} = n^k$ ∎

Theorem: Let n and k be positive integers such that $n \geq k$. Then the number of k-length sequence one can form over an n-element set in which no element is used more than once is,

$$n.(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Proof: Easy. Hence, skipped.

Choosing Subsets of a Set

Definition: The number of k-element subsets of $[n] = \{1,2,\ldots,n\}$ is denoted by $\binom{n}{k}$ and read as "n choose k".

Theorem: For all non-negative integers $k \leq n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof: Selecting a k-length sequence from the n element set can be done in $\frac{n!}{(n-k)!}$ ways.

Every k-element subset occurs $k!$ times among these sequences.

$\therefore$ the number of k-element subsets is $\frac{1}{k!}$ times the number of k-length sequences. ■

Choosing Subsets of a Set

Theorem: For all non-negative integers $k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: We will prove it the longer way...

Let A = set of all k element subsets of $[n]$, and $B =$ set of all $n - k$ element subsets.

Define a bijection $f : A \to B$.

$$f(x) = \text{complement of } x \text{ in } [n].$$

Therefore, $|A| = |B|$. Hence, $\binom{n}{k} = \binom{n}{n-k}$.

■

Choosing Multisubsets of a Set

Definition: A subset of a set that is a multiset is called **multisubset**.

For instance, $\{1,2,3\}$, $\{2,2,2\}$, $\{1,1,1\}$, etc., are 3-element multisubsets of the set $[10]$.

Theorem: The number of k-element multisubsets of $[n]$ is $\binom{n+k-1}{k}$.

Proof: Let,

$$A = \text{set of } k\text{-element multisubsets of } [n]$$

$$B = \text{set of } k\text{-element subsets of } [n+k-1]$$

Define a bijection $f : A \to B$.

$$f(\{x_1, x_2, x_3, \ldots, x_k\}) \; = \; \{x_1, \; x_2 + 1, x_3 + 2, \ldots, x_k + (k-1)\}, \text{ where } x_i \leq x_{i+1}.$$

Choosing Multisubsets of a Set

Proving range of f is B: Let $\{x_1, x_2, x_3, \ldots, x_k\} \in A$, such that $x_1 \leq x_2 \leq \ldots \leq x_k$.

Consider $f(\{x_1, x_2, x_3, \ldots, x_k\}) = \{x_1,\ x_2 + 1, x_3 + 2, \ldots, x_k + (k-1)\}$

Then,

- $x_i \leq x_{i+1} \implies x_i + (i-1) < x_{i+1} + i$
- $1 \leq x_1$
- $x_k \leq n + k - 1$

Proving f is one-to-one: DIY.

Proving f is onto: Let $\{y_1, y_2, \ldots, y_k\} \in B$ such that $y_1 < y_2 < \ldots < y_k$.

Then, $\{y_1, y_2 - 1, y_3 - 2, \ldots, y_k - (k-1)\} \in A$ can be proven the similar way.

Of course, $f(\{y_1, y_2 - 1, y_3 - 2, \ldots, y_k - (k-1)\}) = \{y_1, y_2, y_3, \ldots, y_k\}$. ■